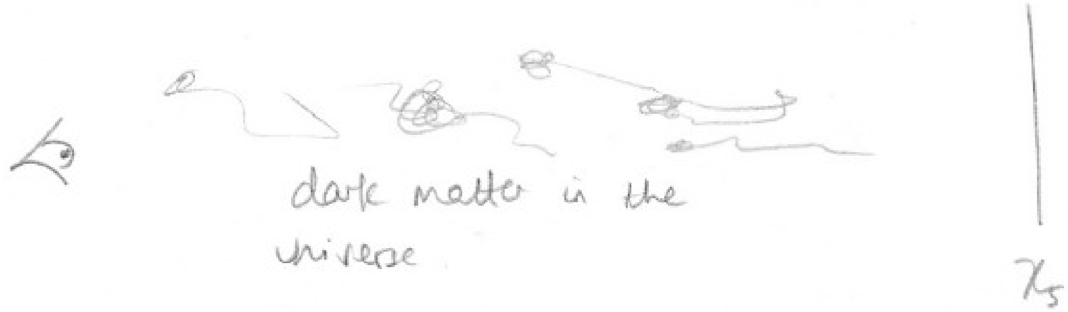
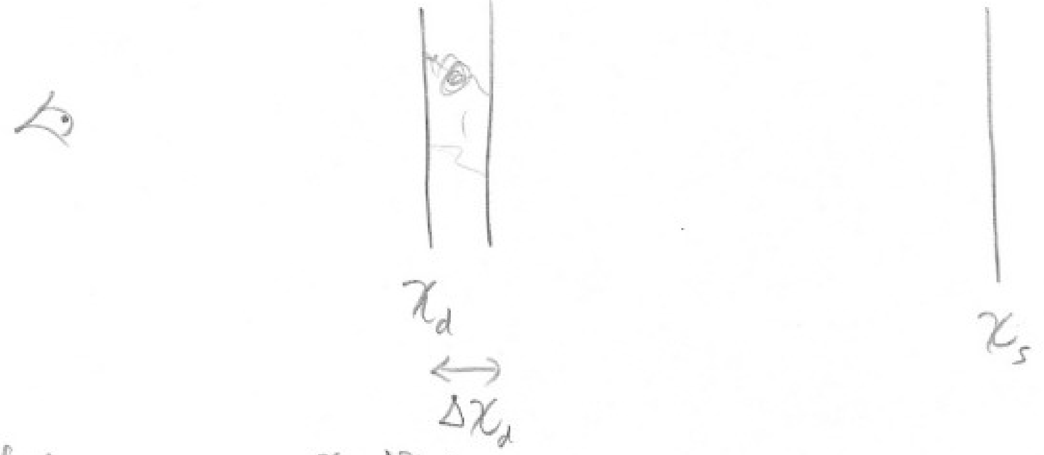
The convergence for a slice through the Universe SLB 1 June 2007



χ = comoving distance

Divide the volume up into slices of thickness $\Delta\chi_d$



surface mass density

$$\Sigma = \int_{\chi_d - \Delta\chi_d/2}^{\chi_d + \Delta\chi_d/2} \rho_{\text{physical}}(\underline{x}^p)\, dx_3^p$$

where $\rho_{\text{physical}}(\underline{x}^p)$ is the physical mass density (ie. mass per physical volume not comoving volume) as a function of physical (not comoving) position.

$$\Rightarrow \quad \Sigma = \int_{\chi_d - \Delta\chi_d/2}^{\chi_d + \Delta\chi_d/2} \rho_{\text{physical}}(\underline{x}^p) \underbrace{\frac{d\chi_d}{(1+z_d)}}_{\text{physical length corresponding to comoving length } d\chi_d}$$

$\rho_{\text{comoving}}(\underline{x})$ is the quantity we more usually deal with in cosmology. It is the mass per unit comoving volume, as a function of comoving position $\underline{x}$

hereafter: $\rho = \rho_{\text{comoving}}$.

The convergence for a slice through the universe   SLB 1 June 2007 (2)

$$\rho_{(\text{comoving})}(\underline{x}) = \frac{\rho_{\text{physical}}(\underline{x}')}{(1+z_d)^3}$$

(since comoving density is mass divided by a big number...)

$$\Rightarrow \quad \Sigma = \int \rho_{(\text{comoving})}(\underline{x}) \, (1+z_d)^3 \, \frac{d\chi_d}{(1+z_d)}$$

$$= \int \rho_{(\text{comoving})}(x) \, d\chi_d \, (1+z_d)^2$$

For cosmology we define $\delta(\underline{x}) \equiv \dfrac{\rho(x) - \bar{\rho}}{\bar{\rho}}$

$$\rightarrow \quad \rho(\underline{x}) = \bar{\rho}\,(\delta(\underline{x}) + 1)$$

where $\bar{\rho} = \rho_{crit}\,\Omega_m = \dfrac{3\,\Omega_m H_0^2}{8\pi G}$

$$\rightarrow \quad \Sigma = \int \frac{3\,\Omega_m H_0^2}{8\pi G} \, (\delta(\underline{x}) + 1) \, (1+z_d)^2 \, d\chi_d$$

$$\rightarrow \quad \kappa \equiv \frac{\Sigma}{\Sigma_{crit}}$$

note this 'critical' density is unrelated to ρ_{crit}!

where $\Sigma_{crit} = \dfrac{c^2}{4\pi G} \dfrac{D_s}{D_d D_{ds}}$

$$\rightarrow \quad \kappa = \int \frac{4\pi G}{c^2} \frac{D_{ds} D_d}{D_s} \frac{3\,\Omega_m H_0^2}{8\pi G} (\delta(\underline{x})+1)(1+z_d)^2 \, d\chi_d$$

$$= \int \frac{3}{2}\,\Omega_m \frac{H_0^2}{c^2} (1+z_d)^2 \frac{D_{ds} D_d}{D_s} \, (\delta(\underline{x}) + 1)\, d\chi_d$$

The convergence for a slice through the universe SLB 1 June 2007 (3)

For a flat universe
$$D_d = \frac{\chi_d}{(1+z_d)}$$
$$D_{ds} = \frac{\chi_s - \chi_d}{(1+z_s)}$$
$$D_s = \frac{\chi_s}{(1+z_s)}$$

$$\Rightarrow \quad \frac{D_{ds} D_d}{D_s} = \frac{(\chi_s - \chi_d)\chi_d}{\chi_s} \frac{1}{(1+z_d)}$$

$$\Rightarrow \quad \kappa = \frac{3}{2} \Omega_m \frac{H_0^2}{c^2} (1+z_d) \frac{(\chi_s - \chi_d)\chi_d}{\chi_s} \int (\delta(\chi) + 1)\, d\chi_d$$

↑ ok to move it here if $\Delta\chi_d$ is small since z_d, χ_d, χ_{ds} are slowly varying.

~~Lots of people write~~

In practice the sources are distributed at all redshifts & with probability distribution $n(\chi_s)$ where $\int n(\chi_s)\, d\chi_s = 1$

So the effective κ is then

$$\kappa = \frac{3}{2} \Omega_m \frac{H_0^2}{c^2} (1+z_d) \int_{\chi_d}^{\infty} n(\chi_s) \frac{(\chi_s - \chi_d)}{\chi_s} \chi_d \, d\chi_s \int (\delta(\chi)+1)\, d\chi_d$$

$$\equiv \quad q(\chi_d) \int (\delta(\chi) + 1)\, d\chi_d \qquad \text{to simplify it}$$

where $$q(\chi) = \frac{3}{2} \Omega_m \frac{H_0^2}{c^2} (1+z(\chi)) \int_{\chi_d}^{\infty} n(\chi_s) \frac{(\chi_s - \chi)}{\chi_s} \chi \, d\chi_s$$